Chapter 8

Annuities, Amortizations and Funding in the Case of Term Structures

8.1. Capital value of annuities in the case of term structures

In Chapter 5 the annuity evaluation, defined as a financial operation for which the amounts do not show any sign inversion, has been made in the case of a flat structure. Only in Chapter 6, regarding annuities formed by loan amortization installments and the management of bond loans, did we consider briefly the case of varying rates.

It is appropriate here to extend the scenario, assuming that such supplies are made in a perfect market, featured by a given term rate structure and then spot prices for goods with delayed delivery, obtained applying discount factors to the forward values of such goods. In a wider context than that of the security market and with the symbols introduced in Chapter 7, if $v(y,z)$ defined in (7.5) is used as a discount factor to apply to the value S_z an asset with purchase in y and delivery at time $z>y$ to have the *spot price* $P_{y,z}$, then

$$P_{y,z} = v(y,z)\, S_z \tag{8.1}$$

while for a transaction at time $x<y<z$, fixing the value defined in (7.16), the *forward price* $P_{x;y,z}$ in y of an asset of value S_z at delivery z is given by

$$P_{x;y,z} = s(x:y,z)\, S_z \tag{8.1'}$$

If the market is perfect (then the property of independence from the transaction time holds true, given by (7.19)), we have $\forall x$: $s(x;y,z) = v(y,z)$, where $P_{x;y,z} = P_{y,z}$. However, the market *coherence* property, defined in Chapter 7, is enough for the developments of this chapter.

Let us consider a complex operation O whose amounts have the same sign and are payable according to a tickler with n dates in a given interval; we have seen already that it is not reductive to assume this tickler is equally spaced[1]. The rates are per period in the case of a given term structure. For simplicity we will refer mainly to annual rate structures and to annual periods, unless otherwise stated, for which what has been said in section 7.5.2 holds.

Then O has a tickler on a time horizon of n years, that can be written: $(T,T+1,....,T+n)$; let us indicate the corresponding amounts with R_0, R_1,...,R_n, assuming them to be all negative and at least one positive. It is known that this operation O is called *annuity,* temporary for n years[2] (R_h are the *installments* of the annuity) and it can never be fair. In an annuity with delayed payments it is with certainty $R_0 = 0$; if the payments are advance, it definitely is $R_n = 0$.

Generalizing the formulation seen in Chapter 5, where we assumed a *flat rate structure*, we can here evaluate the annuity O at any time on the basis of a *discrete term* according to what was specified in section 7.5.1. It is clear that the results that will be obtained in this chapter – where we generalize those obtained in Chapters 5 and 6 considering annuities, amortizations of shared and unshared loans and funding, evaluated on the basis of varying rates according to term structures – are meaningful only if we can assume that the rate structure, introduced at the starting time, remains valid for the whole time horizon of the considered operation. On the contrary a periodic adjustment of the structure is necessary to evaluate the pro-reserves.

It is convenient here to reinterpret the spot and forward prices defined in Chapter 7 also in terms of *discount factors* for the evaluation. In addition, recalling an observation introduced in section 7.5.3, it is convenient, also with discrete ticklers, to obtain the term structure following from a function (integrable) of *instantaneous intensity* $\delta(x,y)$. We suppose that this intensity on the considered time horizon (assuming the reflexivity and symmetry, with the meaning specified in Chapter 2, and in particular cases also the strong decomposability) holds.

1 This topic has been discussed in section 7.5.1.

2 We could consider the case of perpetuities as the limit case for $n \to \infty$, but it is hard to introduce a term structure on an infinite interval.

For known results, generalizing the formulae from (7.42) to (7.47) we find the following *spot* and *forward* elements of the structure in the discrete scheme, i.e. with the constraints (7.24):

- *the spot present value*

$$v(T,T+k) = a\,(T+k,T) = 1/m(T,T+k) = e^{-\int_T^{T+k}\delta(T,\tau)d\tau} \tag{8.2}$$

- *the delayed interest spot rate (on the unitary base)*

$$i(T,T+k) = m(T,T+k)^{1/k} -1 = v(T,T+k)^{-1/k} -1 = \left(e^{\int_T^{T+k}\delta(T,\tau)d\tau}\right)^{1/k} -1 \tag{8.3}$$

- *the spot return at maturity*

$$\phi(T,T+k) = \left(\int_T^{T+k}\delta(T,\tau)d\tau\right)/k \tag{8.4}$$

- *the advance interest spot rate (on the unitary base)*

$$d(T,T+k) = 1 - v(T,T+k)^{1/k} = 1 - \left(e^{-\int_T^{T+k}\delta(T,\tau)d\tau}\right)^{1/k} \tag{8.5}$$ [3]

- *the forward present value*

$$s(T;T+h,T+k) = v(T,T+k)/\,v(T,T+h) = e^{-\int_{T+h}^{T+k}\delta(T,\tau)d\tau} \tag{8.2'}$$

- *the delayed interest forward rate (on the unitary base)*

$$i(T;T+h,T+k) = s(T,T+h,T+k)^{-1/(k-h)}-1 = \left(e^{\int_{T+h}^{T+k}\delta(T,\tau)d\tau}\right)^{1/(k-h)} -1 \tag{8.3'}$$

- *the forward return at maturity*

$$\phi(T;T+h,T+k) = \left(\int_{T+h}^{T+k}\delta(T,\tau)d\tau\right)/\,(k-h) \tag{8.4'}$$

- *the advance interest forward rate (on unitary base)*

$$d(T;T+h,T+k) = 1 - s(T;T+h,T+k)^{1/(k-h)} = 1 - \left(e^{-\int_{T+h}^{T+k}\delta(T,\tau)d\tau}\right)^{1/(k-h)} \tag{8.5'}$$

Generalizing what was seen in Chapter 4, where a flat rate structure is considered, the value in T^* of O is called *capital value* of the annuity; or, more

3 From (8.3) and (8.5) it follows that *i(T,T+h)* and *d(T,T+h)* are (mean) annual rates in the interval from *T* to *T+h*.

precisely: *initial value* or *present value of prompt annuity* if $T^* = T$; *present value of deferred annuity* if $T^* < T$; *final value* if $T^* = T+n$. The *present value* (then with $T^* \leq T$) is expressed by

$$V_a(T^*) = \sum_{h=0}^{n} R_h\, v(T^*, T+h) = \sum_{h=0}^{n} R_h\, [1 - d(T^*, T+h)]^{T+h-T^*} \tag{8.6}$$

The *final value* is expressed by

$$V_f(T^*) = \sum_{h=0}^{n} R_h\, m(T+h, T^*) = \sum_{h=0}^{n} R_h\, [1 + i(T+h, T+n)]^{n-h} \tag{8.7}$$

Relations (8.6) and (8.7) take into account the spot rates that are valid on the respective payment time.

On the basis of considerations discussed for financial operations (see Chapter 4), the operations $O'' \cup (-W_a(T^*), T^*)$ with $T^* \leq T$ and $O'' \cup (-W_f(T^*), T^*)$ with $T^* = T+n$ are fair according to the adopted financial laws.

Clearly if T^* and T are integers we can give an *integer term structure* of rates and values for n unitary periods. Let us assume such a position, adopting the formulations (7.25) and the positions from (7.26) to (7.39) and indicating the times (that are also the distances with sign from the origin 0, that we choose as the reference time for the rates structure) with lower case letters.

Given the above, the initial value of the prompt annuity on the horizon $[0,n]$, obtainable from (8.6) with $T^* = 0$, is given by

$$V_a(0) = \sum_{k=0}^{n} R_k v_k = \sum_{k=0}^{n} R_k\, (1+i_k)^{-k} \tag{8.6'}$$

or, according of the forward rates in the term structure[4],

$$V_a(0) = \sum_{k=0}^{n} R_k \prod_{r=1}^{k} 1/(1+i_{r-1,r}) \tag{8.6''}$$

Example 8.1

Using n=5, let us assign on the market at time 0 the structured system of interest rates on annual periods:

$i_{0,1} = 0.0418$; $i_{1,2} = 0.0461$; $i_{2,3} = 0.0524$; $i_{3,4} = 0.0485$; $i_{4,5} = 0.0432$

4 In (8.6'') when k=0, the product is put equal to 1. The same observation is valid for some other following formula.

Let us consider an annual annuity-immediate on the horizon (0,5), formed by the monetary amounts

$$R_1 = 1{,}250\ ; \quad R_2 = 1{,}389\ ; \quad R_3 = 1{,}450\ ; \quad R_4 = 1{,}310\ ; \quad R_5 = 1{,}100$$

By using such data applied directly by (8.6") we obtain the present value of the prompt annuity

$$V_a(0) = R_1(1.0418)^{-1} + R_2(1.0418\cdot 1.0461)^{-1} + R_3(1.0418\cdot 1.0461\cdot 1.0524)^{-1} +$$
$$+ R_4(1.0418\cdot 1.0461\cdot 1.0524\cdot 1.0485)^{-1} + R_5(1.0418\cdot 1.0461\cdot 1.0524\cdot 1.0485\cdot 1.0432)^{-1} =$$
$$= 5{,}704.78.$$

If the spot rates i_k , that we find by coherence with the previous ones, were given directly, resulting in

$$i_1 = 0.0418;\ i_2 = 0.0439;\ i_3 = 0.0468;\ i_4 = 0.0472;\ i_5 = 0.0464$$

(from which the prices would be

$$v_1 = 0.59877\ ;\ v_2 = 0.917577\ ;\ v_3 = 0.871890;\ v_4 = 0.831559\ ;\ v_5 = 0.797123\ )$$

we could apply (8.6') still obtaining 5704.78.

To obtain the initial value of annuities-deferred of m years, assuming a rate structure for $m+n$ years, it is enough to multiply $V_a(0)$ by a discount factor relative to the deferment. Such a discount factor is given by v_m and therefore the present value in 0 of the annuity-deferred by m years with installments R_k at times $m+k$, ($k = 0,\dots,n$), is expressed, according to the rate structure, by

$$_{m/}V_a(0) = \prod_{r=1}^{m}(1+i_{r-1,r})^{-1}\sum_{k=0}^{n} R_k \prod_{r=1}^{k}(1+i_{m+r-1,m+r})^{-1} \qquad (8.8)$$

Example 8.2

Let us consider at 0 the annual annuity-due, deferred by 3 years, consisting of 3 payments:

$$R_3 = 10{,}500\ ;\ R_4 = 11{,}600\ ;\ R_5 = 40{,}300$$

The varying rates structure

$$i_{0,1} = 0.030\ ; \quad i_{1,2} = 0.035\ ; \quad i_{2,3} = 0.040; \quad i_{3,4} = 0.037\ ; \quad i_{4,5} = 0.034$$

The present value of the deferred annuity is then

$$_{m/}V_a(0) = (1.030\cdot 1.035\cdot 1.040)^{-1}\,[10500 + 11600\,(1.037)^{-1} +$$
$$+ 40300\,(1.037\cdot 1.034)^{-1}] = 53459.71$$

Using the same hypothesis, the final value of the annuity at time n is expressed as a function of the accumulation forward factor $r_{k,n} = r(0;k,n)$ by

$$V_f(n) = \sum_{k=0}^{n} R_k \, r_{k,n} = \sum_{k=0}^{n} R_k \, (1+i_{k,n})^{n-k} \tag{8.7'}$$

or, as a function of the forward rates in the term structure

$$V_f(n) = \sum_{k=0}^{n} R_k \prod_{r=k+1}^{n} (1+i_{r-1,r}) \tag{8.7''}$$

Relations (8.6"), (8.8) and (8.7") directly use the varying rates that come from the market conditions.

Example 8.3

On a three year interval, assuming the semester as a time unit, let us assign the (spot and forward) interest rates structure on a semiannual base as well as the semiannual annuity-immediate, whose payments are

R_1 = 8,500 ; R_2 = 9,250 ; R_3 = 8,620 ; R_4 = 12,628 ; R_5 = 4,644 ; R_6 = 6,240

Let us find the final value, extracting from the structure the following uniperiod forward rates:

$i_{1,6}$ = 0.0490 ; $i_{2,6}$ = 0.0475 ; $i_{3,6}$ = 0.0465; $i_{4,6}$ = 0.0450 ; $i_{5,6}$ = 0.0445

The final value of the annuity is then

$$V_f(6) = 8500 \cdot 1.0490^5 + 9250 \cdot 1.0475^4 + 8620 \cdot 1.0465^3 + 12628 \cdot 1.0450^2 +$$
$$+ 4644 \cdot 1.0445 + 6240 = 56693.59$$

8.2. Amortizations in the case of term structures

Extending what has been said in Chapter 6, with the positions and symbols defined above, we can develop the theory of amortizations assuming a financial law obtained according to a term structure. To remain closer to the financial market behavior, *we will not assume the independence of the structure from the referring time.*

The amortization with varying rates has been considered in section 6.5 only for the case of uniperiod *spot* rates. It is appropriate to refer to this scheme when it is not realistic to assume the validity of the structure for the whole length of the amortization. For the opposite assumption, we assume then the variability of the rates according to a more general rate structure scheme fixed at time 0 of the loan inception, where the amortization flow is, technically, an “annuity” for which the initial value calculated on the basis of such structure is equal to the debt to be amortized. The amortization installments are mostly periodic, thus annual, semiannual, etc.

We will refer to cases of the annual period; for a period of a different length it is sufficient to change the unit of measure. In the presence of pre-amortization, it is sufficient to refer to the true amortization interval, in which the principal repayments are paid, following the one in which only interest is paid.

However, in the case of varying installments (as far as the outstanding loan balance will not increase with time) it is obvious that, if the initial debt, the length of the amortization and the rate structure valid in the same interval are given, infinite solutions exist for the installments vector used to amortize the debt. This means that, from the lender point of view[5], the payment of the lent amount and the encashment of such installments form altogether a fair operation in relation to the given rate structure. Instead, if the installment invariance is postulated, then the financial equilibrium equation gives the constant installment as the only unknown.

We will limit our analysis to the following cases of amortization:

- the general case of varying installments;
- the particular case of constant installments;
- the particular case of constant principal repayments;
- the case of life amortization.

8.2.1. *Amortization with varying installments*

Let there be the initial debt S to be amortized in n unitary periods (in particular annual), according to a term structure given at initial time 0, for which formulations (7.25) and relations (7.26) and (7.39) hold true. The equivalence between debt and vector $\{R_k\}$, (k=0,1,....,n), of the installments paid at the assigned dates gives the constraint that defines a solution $\{R_k\}$ for the amortization. This is found from (8.6') or (8.6") putting $V_a(0) = S$. We then obtain the following relation, that is the constraint of *financial closure* between debt and amortization installments:

5 We highlight the lender point of view that is usually the “stronger party” in the contract. It is clear that the borrower adapts himself to the conditions fixed by the lender and accepts the contract if, on the basis of his utility and his alternative possibilities on the market, he considers convenient the conditions offered by the lender.

$$S = \sum_{k=0}^{n} R_k \, v_k = \sum_{k=0}^{n} R_k \, (1+i_k)^{-k} = \sum_{k=0}^{n} R_k \prod_{r=1}^{k} (1+i_{r-1,r})^{-1} \quad (8.9)$$

where v_k and i_k correspond to the given $i_{h,k}$ on the basis of the known relations. Therefore, given S, n and the term structure, a vector of components R_k, (k=0,1,....,n), that satisfies (8.9) gives a solution to the amortization[6]. As already mentioned, some restrictions on the arbitrariness of $\{R_k\}$ follow from the eventual constraint of the outstanding loan balance not increasing in time. In addition, we talk about amortizations

- with *delayed* installments, if $R_0 = 0$;
- with *advance* installments, if $R_n = 0$.

Due to (8.9) it is obvious that $\{i_{r\text{-}1,r}\}$ gives a rate structure of cost for the borrower, i.e. a generalized internal rate of return (GIRR) in the sense set out in section 4.4.2.

As happens for a constant rate, each installment is divided into *principal repaid* and *interest paid*, and can, by convention, be paid by the debtor in a delayed or advance way: if both are paid delayed or advance, one has amortization with *delayed* or, respectively, *advance* installments.[7]

Amortization with delayed installments

The development of the delayed amortization schedule includes the interest amounts I_k , the principal repayments C_k and the outstanding balances D_k at time k, that follows from the following equations system

$$(k = 1, \dots, n) \begin{cases} I_k = D_{k-1} \, i_{k-1,k} \\ D_k = D_{k-1} - C_k \\ R_k = I_k + C_k \end{cases} \quad (8.10)$$

6 It is obvious that independence from the transaction time, an assumption that in fact is not very realistic, could lead to the equality between forward and spot rates, i.e.: $\{i(0;h;k)\} = \{i(h,k)\}$, $\forall h,k$, from which it would follow that the outstanding amounts and outstanding loan balances expressed by (8.11) or (8.11') would coincide. However, if the amortization is agreed with indexed interests on the basis of the resulting market rates (where the agreed schedule in 0 on the basis of the term structure at this time is only an estimated calculatation), if the market does not behave as a "perfect market", the inequality $i(r\text{-}1,r) \neq i(0;\, r\text{-}1.r)$ can follow at the r^{th} year with possible differences between estimated and final balance. In such a case some adjustments are needed.

7 We do not consider here the case of advance interest payments and delayed principal repayments – sometimes used in the past in the particular case of *German amortization* (see Chapter 6) – because this scheme is not used often.

with the initial condition $D_0 = S$. From here follows:

Theorem. For a delayed amortization in the case of a term structure, from the recursive relations (8.10) *for the outstanding loan balances we obtain*

$$D_h = \sum_{k=h+1}^{n} R_k\, s_{h,k} = \sum_{k=h+1}^{n} R_k \prod_{r=h+1}^{k} ([1+i_{r-1,r})^{-1} \tag{8.11}$$

that extends to the case $h>0$ *the relation of financial closure* (8.9) *with* $R_0 = 0$ (case $h=0$, $S=D_0$)*. Therefore,* $\forall k$ *the exchange of the outstanding balance in k with the outstanding installment flow at their respective due dates is fair, i.e. on the basis of the given structure the outstanding balance coincides with the pro-reserve.*

Proof. Proceeding by induction, let us verify (8.11) for $h=1$. Since D_0 is given by (8.9) with $R_0 = 0$ and taking into account (7.28'''), we obtain:

$I_1 = i_1 D_0 = d_1(R_1 + \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r})$;

$C_1 = R_1 - I_1 = R_1 v_1 - d_1 \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}$

$D_1 = D_0 - C_1 = (R_1 v_1 + \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}) - (R_1 v_1 - d_1 \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}] =$

$= \sum_{k=2}^{n} R_k (\prod_{r=1}^{k} s_{r-1,r} + d_1 \prod_{r=2}^{k} s_{r-1,r}) = \sum_{k=2}^{n} R_k \prod_{r-2}^{k} s_{r-1,r}$,

because $v_1+d_1 = 1$.

Let us then verify that if (8.11) is true for $h\geq 1$, it is also true for $h+1$. Recalling (7.39'), if $D_h = \sum_{k=h+1}^{n} R_k \prod_{r=h+1}^{k} s_{r-1,r}$ then $I_{h+1} = i_{h,h+1} D_h = d_{h,h+1}(R_{h+1} + \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r})$;

$C_{h+1} = R_{h+1} - I_{h+1} = R_{h+1} s_{h,h+1} - d_{h,h+1} \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r}$; then $D_{h+1} =$ $D_h - C_{h+1} = (R_{h+1} s_{h,h+1} + \sum_{k=h+2}^{n} R_k \prod_{r=h+1}^{k} s_{r-1,r}) - (R_{h+1} s_{h,h+1} -$ $d_{h,h+1} \sum_{k=h+2}^{n} R_k \prod_{r=h+2}^{k} s_{r-1,r}) = \sum_{k=2}^{n} R_k \prod_{r=2}^{k} s_{r-1,r}$,

because $s_{h,h+1} + d_{h,h+1} = 1$.

Example 8.4

Let there be underwritten at time 0 a loan contract for €86000 to be amortized with delayed varying annual installments over 10 years, on the basis of a term structure expressed by spot factors v_k (k=1,...,10) fixed at 0, from which by means of (7.31) we find the forward rates to apply annually. These rates are indicated in the 2nd column of the following *Excel* amortization schedule, the quantities of which follow from recursive relations (8.10) starting from a given principal repaid C_k indicated in the 3rd column, according to a choice that gives higher payments in the central years. On the contrary, if we assigns the installments, satisfying (8.9) as in column 5, are used, obtaining the outstanding balance by (8.11), we find the interest paid (4th column) and then the principal repaid (3rd column).

Debt = 86000 *Length* = 10

Year	Forward rate	Principal repaid	Interest amount	Installment	Outstanding balance
K	$I_{k-1,k}$	C_k	I_k	R_k	D_k
0					86000.00
1	0.050	5000.00	4300.00	9300.00	81000.00
2	0.048	6000.00	3888.00	9888.00	75000.00
3	0.046	7000.00	3450.00	10450.00	68000.00
4	0.044	8000.00	2992.00	10992.00	60000.00
5	0.042	12000.00	2520.00	14520.00	48000.00
6	0.040	15000.00	1920.00	16920.00	33000.00
7	0.043	12000.00	1419.00	13419.00	21000.00
8	0.046	8000.00	966.00	8966.00	13000.00
9	0.049	7000.00	637.00	7637.00	6000.00
10	0.052	6000.00	312.00	6312.00	0.00
		86000.00			

Table 8.1. *Example of delayed amortization*

The *Excel* instructions are as follows. The first three rows are for data and titles; C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty. 5th to 14th rows:

column A (years):	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	from B5 to B14: insert data;
column C (principal repayments):	from C5 to C14: insert data with constraint: "SUM(C5:C14)"= C1 in C15 (to control);
column D (interest payments):	D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (installments):	E5:= C5+D5; copy E5, then paste on E6 to E14;
column F (outstanding balances):	F5:= F4-C5; copy F5, then paste on F6 to F14.

Amortization with advance installments

The development of the advance amortization schedule includes the installments $\ddot{R}_k$ made by the interest payments $\ddot{I}_k$ and by the principal repayments $\ddot{C}_k$, payable for the $(k+1)^{\text{th}}$ period soon after the integer time k, and also the outstanding balances D_k at time k, that results from the following equation system

$$(k = 0,...,n-1)\begin{cases} \ddot{I}_k = D_{k+1}\ d(0;k,k+1) \\ D_{k+1} = D_k - \ddot{C}_k \\ \ddot{R}_k = \ddot{I}_k + \ddot{C}_k \end{cases} \tag{8.10'}$$

using the initial constraint $D_0 = S$. Owing to (8.10') we can deduce the following theorem.

Theorem. For an advance amortization in the case of a rates term structure, from the recursive relations (8.10') *we obtain for the outstanding balances the expression*

$$D_h = \sum_{k=h}^{n-1} \ddot{R}_k\ s_{h,k} = \sum_{k=h}^{n-1} \ddot{R}_k \prod_{r=h+1}^{k} (1 - d_{r-1,r}) \tag{8.11'}$$

This formula extends to the case h>0 *the relation of financial closure* (8.9) *with* R_n=0 (case h=0, S=D_0). *Therefore,* ∀*h the exchange of the outstanding balance in h with the flow of outstanding installments at their due dates is fair, i.e. in the case of this structure the outstanding balances coincide with the pro-reserve.*

The proof of this theorem, that gives rise to (8.11'), proceeds by induction analogously to the one that leads to (8.11), taking into account the identities

$$1 - d_{r-1,r} = s_{r-1,r} = (1 + i_{r-1,r})^{-1}$$

which give a value equal to v_r/v_{r-1} in the perfect market hypothesis. Although for sake of brevity it is omitted, we would say that in the induction it is convenient to proceed backwards, i.e. verifying (8.11') for h=n-1 and proving that if it holds true for an index h (with 1≤h≤n-1), it is also true for h-1.

Example 8.5

Let us consider again Example. 8.4 assuming a loan for the same amounts, length and distribution of principal repayments, but advance installments and then advance forward rates $d_{k-1,k}$, choosing those equivalent to the delayed rates in Example 8.4. By working in *Excel* we easily obtain the following table.

Debt = 86000.00 *Length* = 10

Year	Delayed forward rate	Advance forward rate	Outstanding balance	Principal Repaid	Interest paid	Installment
k	$i_{k-1,k}$	$d_{k-1,k}$	D_k	ant. C_k	ant. I_k	ant. R_k
0			86000.00	5000.00	3857.14	8857.14
1	0.050	0.047619	81000.00	6000.00	3435.11	9435.11
2	0.048	0.045802	75000.00	7000.00	2990.44	9990.44
3	0.046	0.043977	68000.00	8000.00	2528.74	10528.74
4	0.044	0.042146	60000.00	12000.00	1934.74	13934.74
5	0.042	0.040307	48000.00	15000.00	1269.23	16269.23
6	0.040	0.038462	33000.00	12000.00	865.77	12865.77
7	0.043	0.041227	21000.00	8000.00	571.70	8571.70
8	0.046	0.043977	13000.00	7000.00	280.27	7280.27
9	0.049	0.046711	6000.00	6000.00	0.00	6000.00
10	0.052	0.049430	0.00			
				86000.00		

Table 8.2. *Example of advance amortization*

The *Excel* instructions are as follows. The first three rows are for data and titles; C1: 86000; E1: 10. 4th to 14th rows:

column A (year)	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (delayed forward rate)	B4 empty; from B5 to B14 insert date;
column C (advance forward rate)	C4:= 1-(1+B5)^-1; copy C4, then paste on C5 to C13;
column D (outstanding loan balance)	D4:= C1; D5:= D4-E4; copy D5, then paste on D6 to D14;
column E (advance principal	E4 to E13 insert data with the constraint "SUM(E4:E14)"=C1 in E15 (check);
column F (advance interest paid)	F4:= D5*C5; copy F4, then paste on F5 to F13;
column G (installment)	G4:= E4+F4; copy G4, then paste on G5 to G13.

Observation

In the delayed amortizations, from system (8.10) the following corollary holds.

Corollary. If we have fairness, each vector $\{R_k\}$ *of delayed amortization installments satisfies, for* k=1,...,n,

$$R_k = D_{k-1}\, i_{k-1,k} + (D_{k-1} - D_k) \tag{8.12}$$

i.e. the following recursive relation holds

$$D_k = D_{k-1}(1 + i_{k-1,k}) - R_k \tag{8.13}$$

Analogously in advance amortizations from system (8.10') we can deduce the following corollary:

Corollary. If we have fairness, each vector $\{\ddot{R}_k\}$ of advance amortization installments, satisfies, for k=0,...,n-1

$$\ddot{R}_k = D_{k+1}\, d_{k,k+1} + (D_k - D_{k+1}) \tag{8.12'}$$

i.e. the following recursive relation

$$D_k = D_{k+1}(1 - d_{k,k+1}) + \ddot{R}_k \tag{8.13'}$$

holds.

Proof. Since, owing to the fairness of this operation, $D_n = 0$ holds true, if in the delayed case we write (8.13) for k=1,...,n, with subsequent substitutions we obtain the relation of financial closure and, writing such relation for k=h+1,....,n, we easily obtain (8.11). Analogously if in the advance case we write (8.13') for k=0,...,n-1, with subsequent substitutions we obtain the relation of financial closure and, writing it for k=h,....,n-1 we easily obtain (8.11').

8.2.2. *Amortization with constant installments*

The conclusions for this major case of refund techniques are obtained from the results in section 8.2.1 using R_k or $\ddot{R}_k$ = constant = R. Therefore, given the initial debt S to be amortized in n periods, according to a given (or assumed) term structure at initial time 0, for which formulations (7.25) and the relations from (7.26) to (7.39) hold true, the installment solution is deduced introducing constraint (8.9). Therefore we obtain the following relation:

Delayed case

Using R_0 = 0, R_k = R, (k=1,...,n), in the financial closure relation (8.9), the installment R is given by

$$R = S / \sum_{k=1}^{n} \prod_{r=1}^{k} (1 + i_{r-1,r})^{-1} \tag{8.14}$$

Recursive relations (8.10) hold, where the outstanding balances D_h are expressed by

$$D = R \sum_{k=h+1}^{n} \prod_{r=h+1}^{k} (1 + i_{r-1,r})^{-1} \tag{8.15}$$

Exercise 8.1

Considering again the loan in Example 8.4 with the data given there, find the amortization schedule under the constraint that the delayed installment is constant.

A. Finding the installment by means of (8.9) we obtain R=10916.95; from here, applying recursively (8.10) with $R_k = R$, we obtain, using *Excel* for the following schedule.

debt = 86000		*length* = 10		*installment*= 10916.95	
Year	Forward rate	Discount factor	Interest paid	Principal repaid	Outstanding balance
K	$i_{k-1,k}$	Π by (8.14)	I_k	C_k	D_k
0		1			86000.00
1	0.050	0.952381	4300.00	6616.95	79383.05
2	0.048	0.908760	3810.39	7106.56	72276.48
3	0.046	0.868796	3324.72	7592.23	64684.25
4	0.044	0.832180	2846.11	8070.84	56613.41
5	0.042	0.798637	2377.76	8539.19	48074.22
6	0.040	0.767920	1922.97	8993.98	39080.24
7	0.043	0.736261	1680.45	9236.50	29843.74
8	0.046	0.703883	1372.81	9544.14	20299.60
9	0.049	0.671003	994.68	9922.27	10377.33
10	0.052	0.637836	539.62	10377.33	0.00
		7.877658			

Table 8.3. *Example of amortization with constant delayed installments*

The *Excel* instructions are as follows. The first three rows are for data, columns titles and one calculation: B1: 86000; D1: 10; F1:= B1/C15. 4^{th} row: A4: 0; C4: 1; G4:= B1; 5^{th} to 15^{th} rows:

column A (years):	A5: = A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	insert forward rates (see Example 8.4);
column C (discount factors):	C5:= C4*(1+B5)^-1; copy C5, then paste on C6 to C14; C15:= SUM(C5:C14).
column D (interest payments):	D5: = F4*B5; copy D5, then paste on D6 to B14.
column E (principal repayments):	E5: = $F$1-D5; copy E5, then paste on E6 to E14.
column F (outstanding balances):	F5: = F4-E5; copy F5, then paste on F6 to F14.
other cells	empty.

Observation

An amortization that keeps constant installments in a varying rate regime is possible only in the case that the rate structure is agreed in 0 (or that the perfect market assumption hold true), as assumed in the previous examples. If these assumptions fail and the amortization proceeds in time in a flexible form on the basis of annual varying spot rates $i_{k-1,k}$ (with complete notation) not predictable in 0 and that will be different from $i_{0;k-1,k}$, then the schedule cannot be fixed in advance and we have to proceed as discussed in section 6.5, point a. We adopted in this section the complete formulation of the rate structure because many contracting times are here considered here.

In particular, we can proceed for subsequent renovations of the contract, calculating the installment and its elements each year that the rate changes (using (6.52) and (6.52')) on the basis of the new rate, the outstanding balance, and the remaining length. This procedure is consistent with the constant installment scheme, because if after the renovation the rate no longer changes, the new installment will remain constant, as can be seen from equation $D_h = R\, a_{\overline{n-h}|i}$.

Example 8.6

Let us give an example of the second procedure, that uses the spot rates $i(k\text{-}1;k\text{-}1,k)$. For an easy comparison, let us use the input data of Example 8.4, obtaining the following *Excel* table.

	Debt = 86000		*Length* = 10		
Year	spot rate	installment	interest paid	principal repaid	outstanding balance
K	*I(k-1; k-1, k)*	*Rk*	*Ik*	*Ck*	*Dk*
0					86000.00
1	0.050	11137.39	4300.00	6837.39	79162.61
2	0.048	11038.42	3799.81	7238.61	71924.00
3	0.046	10948.97	3308.50	7640.46	64283.53
4	0.044	10869.12	2828.48	8040.65	56242.89
5	0.042	10798.96	2362.20	8436.76	47806.13
6	0.040	10738.55	1912.25	8826.31	38979.82
7	0.043	10814.58	1676.13	9138.45	29841.37
8	0.046	10875.97	1372.70	9503.27	20338.10
9	0.049	10922.44	996.57	9925.87	10412.24
10	0.052	10953.67	541.44	10412.24	0.00

Table 8.4. *Example of delayed amortization with spot rates*

The *Excel* instructions are as follows. The first three rows are for data and columns titles; C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty; 5th to 15th rows:

column A (years):	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rates):	insert data from B5 to B14;
column C (installments):	C5:= F4*B5/(1-(1+B5)^-($E$1+1-A5)); copy C5, then paste on C6 to C14;
column D (interest payments):	D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (principal repayments):	E5:= C5-D5; copy E5, then paste on E6 to E14;
column F (outstanding balances):	F5:= F4-E5; copy F5, then paste on F6 to F14.

Advance case

Using $R_n = 0$, $R_k = \ddot{R}$, (k=0,...,n-1), in the relation of financial closure (8.9), the installment $\ddot{R}$ is given by

$$\ddot{R} = S \,/ \left[1 + \sum_{k=1}^{n-1} \prod_{r=1}^{k} (1\text{-}d_{r-1,r})\right] \tag{8.14'}$$

Recursive relations (8.10') hold true, where the outstanding balances D_h are expressed by

$$D_h \;=\; \ddot{R} = \;\left[1 + \sum_{k=h+1}^{n-1} \prod_{r=h+1}^{k} (1 - d_{r-1,r})\right] \tag{8.15'}$$

Exercise 8.2

Let us consider a loan of €45,000 with varying rates, a length of 5 years and forward rates, fixed when the contract is signed. Calculate the amortization schedule, where the rates are specified and where the advance installments are constant.

A. To calculate the installment, apply relation (8.14') and for the principal and interest payments (that cannot be calculated starting form the initial debt) we first have to calculate the outstanding balances at the intermediate integer times by means of (8.15') using the identity:

$$\prod_{r=h+1}^{k} (1 - d_{r-1,r}) = \;\prod_{r=1}^{k} (1 - d_{r-1,r}) / \prod_{r=1}^{h} (1 - d_{r-1,r})$$

We then take into account the 2nd of (8.10') for the principal repayments and the 1st and 3rd of (8.10') for the interest paid. Proceeding with *Excel*, we obtain the following schedule with two sections, where the second is an instrument to calculate on single columns the outstanding balances.

	Debt = 45000.00		*Length* = 6		*Installment*= 9865.56	
Year	Delayed forward rate	Advance forward rate	Spot discount factor	Outstanding balance	Principal repaid	Interest paid
k	$i_{k-1.k}$	$d_{k-1.k}$	Π by(8.14')	D_k	*Ant* C_k	*Ant* I_k
0			1	45000.00	8108.84	1756.72
1	0.050	0.047619	0.952381	36891.16	8568.33	1297.23
2	0.048	0.045802	0.908760	28322.82	9016.53	849.03
3	0.046	0.043977	0.868796	19306.29	9440.73	424.83
4	0.045	0.043062	0.831384	9865.56	9865.56	0.00
5	0.047	0.044890		0.00		
			4.561321		45000.00	

Year	Outstanding balance 1	Outstanding balance. 2	Outstanding balance 3	Outstanding balance.4	Outstanding balance 5
k	D_1	D_2	D_3	D_4	D_5
1	1.000000				
2	0.954198	1.000000			
3	0.912236	0.956023	1.000000		
4	0.872953	0.914854	0.956938	1.000000	
	36891.16	28322.82	19306.29	9865.56	0.00

Table 8.5. *Example of amortization with constant advance installments*

The *Excel* instructions are as follows.

1st sector. C1: 45000; E1: 6; G1:= C1/D10; other cells: empty; 2nd and 3rd rows for titles; 4th to 10th rows:

column A (year):	A4: 0; A5:= A4+1; copy A5, then paste on A6 to A9;
column B (delayed forward rate):	B4 empty; insert data from B5 to B9;
column C (advance. forward rate):	C4 empty; C5:= 1-(1+B5)^-1; copy C5, then paste on C6 to C9;
column D (discount factor 0-k):	D4: 1; D5:= D4(*(1-C5); copy D5, then paste on D6 to D8; D9 empty; D10:= SUM(D4:D9);
column E (outstanding debt):	E4:= C1; E5:= B19; E6:= C19; E7:= D19; E8:= E19; E9:= F19
column F (principal repaid):	F4: E4-E5; copy F4, then paste on F5 to F8;
column G (interest paid):	G4:= $G$1-F4; copy G4, then paste on G5 to G8.

2nd sector. 13th and 14th rows for titles; 15th to 19th rows:

column A (year):	A15: 1; A16:= A15+1; copy A16, then paste on A17 to A18; A19 empty;
column B (outstanding balance 1):	B15:= D5/D$5; copy B15, then paste on B16 to B18; B19:= $G$1*SUM($B$15:$B$18);
column C (outstanding balance 2):	C15 empty; C16:= D6/D$6; copy C16, then paste on C17 to C18; C19:= $G$1*SUM($C$16:$C$18);
column D (outstanding balance 3):	D15,D16 empty; D17:= D7/D$7; copy D17, then paste on D18; D19:= $G$1*SUM($D$17:$D$18);
column E (outstanding balance 4)	E15,E16,E17 empty; E18:= E8/E$8; copy E18, then paste on E18; E19:= $G$1*$E$18;
column F (outstanding balance 5)	F15,F16,F17,F18 empty; F19:= $G$1-F8 .

8.2.3. *Amortization with constant principal repayments*

In this case, if the structure of the per period forward rates $\{i_{h,k}\}$ is given, according to the installment due dates on the time interval from 0 to n for the debt S to be amortized, the calculation of such installments gives a unique solution, in the following way.

First of all, the constant principal repaid of the n installments is calculated, which is simply S/n. This implies that the outstanding balances decrease in arithmetic progression with ratio S/n; then after h payments we have an outstanding balance of $S(n\text{-}h)/n$.

For each period the interest rate is found from the vector $\{i_{k\text{-}1,k}\}$, (k=1....,n) and then the installments R_k are

– *in the delayed case:*

$$R_0 = 0\ ;\ R_k = \frac{S}{n}\ [1 + (n - k + 1)\ i_{k\text{-}1,k}]\ \ ,(k = 1,...,n) \tag{8.16}$$

– *in the advance case:*

$$\ddot{R}_k = \frac{S}{n}\,[1 + (n - k - 1)\,d_{k,k+1}]\ \ ,(k = 0,...,n-1)\ ;\ \ddot{R}_n = 0 \tag{8.16'}$$

Exercise 8.3

Let us consider again the problem of amortization and the data used in Example 8.4, but now applying the method with constant principal repayments. Using (8.16) we obtain the following *Excel* table.

Debt = 86000 *Length* = 10

Year	Forward rate	Principal repaid	Interest paid	Installment	Outstanding balance
k	$i_{k-1.k}$	C_k	I_k	R_k	D_k
0					86000.00
1	0.050	8600.00	4300.00	12900.00	77400.00
2	0.048	8600.00	3715.20	12315.20	68800.00
3	0.046	8600.00	3164.80	11764.80	60200.00
4	0.044	8600.00	2648.80	11248.80	51600.00
5	0.042	8600.00	2167.20	10767.20	43000.00
6	0.040	8600.00	1720.00	10320.00	34400.00
7	0.043	8600.00	1479.20	10079.20	25800.00
8	0.046	8600.00	1186.80	9786.80	17200.00
9	0.049	8600.00	842.80	9442.80	8600.00
10	0.052	8600.00	447.20	9047.20	0.00

Table 8.6. *Example of amortization with constant principal repayments*

The *Excel* instructions are as follows. The first three rows are for data and titles. C1: 86000; E1: 10. 4th row: A4: 0; F4:= C1; other cells: empty. 5th to 14th rows:

column A (years):	A5:= A4+1; copy A5, then paste on A6 to A14;
column B (forward rate):	B5 to B14: insert data;
column C (principal repaid):	C5:= C$1/E$1; copy C, then paste on C6 to C14;
column D (interest paid):	D5:= F4*B5; copy D5, then paste on D6 to D14;
column E (installment):	E5:= C5+D5; copy E5, then paste on E6 to E14;
column F (outstanding balance):	F5:= F4-C5; copy E5, then paste on F6 to F14.

8.2.4. *Life amortization*

Having fully described this actuarial operation in section 6.3, we limit ourselves here to briefly considering the variations linked to the introduction into a scheme of *advance life amortization* of a discrete term structure that can be identified by a uniperiod forward rates $\{i_{r\text{-}1,r}\}$ agreed at time 0, indicating with $*$ the quantities that depend on it.

Let S be the debt of the annual loan; n the length in years; $\{i_{r\text{-}1,r}\}$ the structure of the adopted rates, that gives rise to a law that generalizes the IRR of the lender-insurer; x the integer age of the borrower at the drawing up of the contract. In addition, let us indicate the actuarial discount factor on the interval $(z,z+1)$ for the borrower with

$$_1E_x^* = \frac{l_{x+z+1}}{l_{x+z}}(1+i_{z,z+1})^{-1} \tag{8.17}$$

The actuarial discount factor on the interval (0,*z*) is given by

$$_zE_x^* = \prod_{r=0}^{z-1} \frac{l_{x+r+1}}{l_{x+r}}(1+i_{r,r+1})^{-1} \tag{8.18}$$

Then $_zE_x^* = \prod_{r=0}^{z-1} {_1E_{x+r}^*}$ holds true. Let us now take into account now that the uniperiod discount forward rates $d_{r\text{-}1,r}$ are linked to the interest forward rates by the relation

$$1 - d_{r-1,r} = (1+i_{r-1,r})^{-1} = s_{r-1,r} \tag{8.19}$$

Thus, the constraint of financial closure on the advance installments $\ddot{\alpha}_z^*$ that generalizes (6.28) is written as (considering (8.18)):

$$\sum_{z=0}^{n-1} \ddot{\alpha}_z^* \, {_zE_x^*} = S \tag{8.20}$$

Proceeding analogously to section 6.3.1, if the *principal repayments* $\ddot{c}_z$ are given under the constraint (6.29) we find the outstanding balances D_z on the basis of the 1st part of (6.32), from which the advance *actuarial interest payments* $\ddot{j}_z^*$ comes, is given by

$$\ddot{j}_z^* = \left[d_{z,z+1} + (1-d_{z,z+1})q_{x+z} \right] D_{z+1} = (1- {_1E_{x+z}^*})D_{z+1}\,, \; z = 0,\dots,n\text{-}1 \tag{8.21}$$

and using (8.21) we obtain the advance *installments*

$$\alpha_z^* = \ddot{c}_z + \ddot{j}_z^* = D_z - {_1E_{x+z}^*} D_{z+1}\,, \quad z = 0,\dots,n\text{-}1 \tag{8.22}$$

If, instead, the installments $\ddot{\alpha}_z^*$ are given subject to (8.20), as far as the outstanding balances the formula

$$D_z = \sum_{k=z}^{n-1} \ddot{\alpha}_k^* \, {_kE_x^*}\,, \quad z = 0,\dots,n\text{-}1 \tag{8.23}$$

that generalizes (6.31) holds true. The values (8.23) thus allow us to calculate $\ddot{c}_z$ using the 1st of (6.32) and $\ddot{j}_z^*$ using (8.21).

If in *z* the technical bases, fixed in 0, are not changed, (8.23) also gives the *prospective reserves* W_Z while the *retrospective reserves* are expressed by

$$M_z = \frac{S - \sum_{k=0}^{z-1} \ddot{\alpha}_k^* \, {}_kE_x^*}{{}_zE_x^*} \quad , \quad z = 1,...,n\text{-}1 \tag{8.24}$$

Exercise 8.4

Using the financial data in Example 8.4, calculate the advance life amortization schedule with the demographic data in Exercise 6.6.

A. On the basis of the advance uniperiod forward rates deducible from the delayed ones, assigned in the following 3rd column, we obtain the required schedule.

	Debt =	86000			*Length =*	10	
Year	Survival table	Forward rate	Actuarial discount factor	Principal repaid	Outstanding balance	Interest paid	Installment
z	l_{42+z}	$i_{z-1.z}$	E^*_{42+z}	C_z	D_z	J_z	α_z
0	96400		0.950682	5000	86000	3994.78	8994.78
1	96228	0.050	0.952394	6000	81000	3570.47	9570.47
2	96046	0.048	0.954062	7000	75000	3123.78	10123.78
3	95849	0.046	0.955676	8000	68000	2659.45	10659.45
4	95631	0.044	0.957234	12000	60000	2052.76	14052.76
5	95386	0.042	0.958776	15000	48000	1360.38	16360.38
6	95112	0.040	0.955708	12000	33000	930.13	12930.13
7	94808	0.043	0.952736	8000	21000	614.44	8614.44
8	94482	0.046	0.949667	7000	13000	302.00	7302.00
9	94123	0.049	0.946763	6000	6000	0.00	6000.00
10	93746	0.052		86000	0		
total					86000		

Table 8.7. *Example of life amortization*

The *Excel* instructions are as follows. The first three rows are for titles and data. C1: 86000; G1: 10. 4th to 14th rows:

column A (year):	A4: 0; A5:= A4+1; copy A5, then paste on A6 to A14;
column B (survival table):	insert data from B4 to B14;
column C (forward rate):	insert data from C5 to C14;
column D (actuarial discount factors):	D4:= B5*(1/(1+C5))/B4; copy D4, then paste on D5 to D13; D14 empty;
column E (principal repaid):	insert data from E4 to E13 with the constraint: "SUM (E4:E13)" = C1, in E15;

column F (outstanding balance): F4:= C1; F5:= F4-E4; copy F5, then paste on F6 to F14;
column G (interest paid): G4:= (1-D4)*F5; copy G4, then paste on G5 to G13; G14 empty;
column H (installment): H4:= E4+G4; copy H4,then paste on H5 to H13; H14 empty.

8.3. Updating of valuations during amortization

We can generalize to the case of varying rates, according to a term structure, the considerations developed in section 6.6 about residual valuations (pro-reserves) of financial operations with rates changed to the initial rates. Such observations were useful about calculations regarding assignments of a credit, firm valuations, etc. with the application of rates used on the market at the time of calculation. If we are talking about residual valuations regarding gradual amortizations, we use *Makeham's formula* (see section 6.6.2).

With reference to the general amortization of a loan drawing up in 0, shown in section 8.2.1, we can calculate at time $t \in \mathcal{N}$ the loan *pro-reserve* W_t , *usufruct* U_t and *bare ownership* P_t. However it is important that such valuations often have to be made according to the term structure given by the market at time t, summarized − using the complete notation, because of plurality of reference times − by $\{i(t;h,k)\},(t \leq h<k)$ that, under the hypothesis of *dependence on valuation time*, differs from that valid at the loan issue, summarized by $\{i(0;h,k)\}$, according to which the installments, interest and principal payments have been calculated.

Let us refer to a delayed amortization (but the changes for the case of advance amortization are easy) and assigning the payments R_k satisfying (8.9) as well as the interest paid I_k and the principal repaid C_k , satisfying the recurrent system (8.10) and then coherent with the structure $\{i(0;h,k)\}$. Then the pro-reserve W_t at time $t \in \mathcal{N}$, valued according to forward rate structure $\{i(t;h,k)\}$ equivalent to that of spot prices $v(t,k)$, is given by

$$W_t = \sum_{k=t+1}^{n} R_k \, v(t,k) = \sum_{k=t+1}^{n} R_k \prod_{r=t+1}^{k} [1+i(t;r\text{-}1,r)]^{-1} \tag{8.25}$$

having considered the constraints between prices and rates, effective in a coherent market. The pro-reserve W_t is the sum of usufruct U_t, the present value of residual interest payments I_k, and bare ownership P_t, present value of residual principal payments C_k , valued according to the updated structure $\{i(t;h,k)\}$. Then we have:

$$U_t = \sum_{k=t+1}^{n} I_k \, v(t,k) \quad ; \quad P_t = \sum_{k=t+1}^{n} C_k \, v(t,k) \tag{8.26}$$

where I_k and C_k are obtained using (8.10).

If the payments subject to constraint (8.9) and the forward rates' structure are agreed in advance, owing to (7.36)

$$U_t = \sum_{k=t+1}^{n} v(t,k)\; i(0;k-1,k) \sum_{u=k}^{n} R_u \; s(0;k-1,u) \tag{8.27}$$

holds for the usufruct;

$$P_t = W_t - U_t = \sum_{k=t+1}^{n} v(t,k)\; \left[R_k - i(0;k-1,k) \sum_{u=k}^{n} R_u \;\; s(0;k-1,u)\right] \tag{8.27'}$$

holds for the bare ownership.

Example 8.7

Let us apply the previous formulae on a delayed amortization with the given principal repaid on a debt of €100.000 and time length 5 years, for valuing pro-reserves, split into usufruct and bare ownership components, in the rate structure hypothesis changing at each end of year.

Using an *Excel* table, in the first part we calculate the delayed amortization schedule plan of € 100.000 in 5 years, having assigned the principals repaid and rate structure. In the second part, recalling relation (8.25) between unit spot prices and forward rates, we obtain pro-reserves as well as usufructs and bare ownerships according to modified rates, using (8.25) and (8.26) under the hypothesis that in each year all the varying rates after the first increase of 0.2%. The obtained pro-reserves can be compared with outstanding loan balances, reminding us that if the rate change does not occur, in each period we should have equality. Carrying out the calculations we obtain the following table.

PART 1

Debt = 100000 *Length* = 5

Year	Forward rate	Principal repaid	Interest paid	Installment	Outstanding balance
K	$i_{k-1,k}$	C_k	I_k	R_k	D_k
0					100000.00
1	0.040	10000.00	4000.00	14000.00	90000.00
2	0.043	20000.00	3870.00	23870.00	70000.00
3	0.046	30000.00	3220.00	33220.00	40000.00
4	0.044	30000.00	1760.00	31760.00	10000.00
5	0.042	10000.00	420.00	10420.00	0.00

PART 2

Calculus of spot rates

Year	Modified forward rate	Spot price	Spot price	Spot price	Spot price
K		$V_{1,k}$	$V_{2,k}$	$V_{3,k}$	$V_{4,k}$
1		1.000000			
2	0.045	0.956938	1.000000		
3	0.048	0.913109	0.954198	1.000000	
4	0.046	0.872953	0.912236	0.956023	1.000000
5	0.044	0.836162	0.873789	0.915731	0.957854

Calculation of pro-reserves, usufructs and bare ownerships

Year	Pro-reserve	Usufruct	Bare ownership
k	W_k	U_k	P_k
1	89613.36	8531.14	81082.21
2	69775.96	5045.05	64730.91
3	39905.20	2067.21	37838.00
4	9980.84	402.30	9578.54

Table 8.8. *Calculation of pro-reserves, usufructs and bare-ownerships*

The *Excel* instructions for the first part are analogous to that specified in Example 8.4 which works out this type of amortization kind. The instructions for the second part are as follows.

12th to 15th rows:	titles
16th to 20th rows:	calculation of unit prices (as discount factors):
column A (year):	A16: 1; A17:= A16+1; copy A17, then paste on A18 to A20;
column B (updated fwd rate):	B16 empty; input of data from B17 to B20;

column C (v(1,k)):	C16: 1; C17:= C16*(1+$B17)^-1; copy C17, then paste on C18 to C20;
column D (v(2,k)):	D16 empty; D17: 1; copy C17, then paste on D18 to D20;
column E (v(3,k)):	E16, E17 empty; E18: 1; copy C17, then paste on E19 to E20;
column F (v(4,k)):	F16, F17, F18 empty; F19: 1; copy C17, then paste on F20.

21th row: empty; 22th to 24th rows: titles.
25th to 28th rows: calculation of pro-reserves, usufructs and bare ownerships:

column B (year):	B25: 1; B26:= B25+1; copy B26, then paste on B27 to B28;

in the following right-side columns we calculate “scalar products between vectors” using *Excel* function “MATR-SUM-PRODUCT” here abreviated as MSP:
column C (pro-reserve = scalar product between installments and prices)
C25 := MSP(E7:E10;C17:C20); C26 := MSP(E8:E10;D18:D20);
C27 := MSP(E9:E10;E19:E20); C28 := MSP(E10;F20);
column D (usufruct = scalar product between interest paid and prices)
D25 := MSP(D7:D10;C17:C20); D26 := MSP(D8:D10;D18:D20);
D27 := MSP(D9:D10;E19:E20); D28 := MSP(D10;F20);
column E (bare ownership = scalar product between principal repaid and prices)
E25 := MSP(C7:C10;C17:C20); E26 := MSP(C8:C10;D18:D20);
E27 := MSP(C9:C10;E19:E20); E28 := MSP(C10;F20).

8.4. Funding in term structure environments

We can generalize the problem already considered in section 6.4, by assigning the equivalence relation between:

– a monetary amount that has to be set up at a given maturity t;

– a concordant payments set, then an annuity, with tickler before t and embedded into a financial structure giving accrued interest, fit to give such an amount at t.

For the sake of simplicity we assume periodic payments as in section 8.1 and for the annuity a horizon of *n* periods (in particular, *n* years). Moreover, let us settle the term structure giving the uniperiod forward immediate rates $\{i_{r-1,r)}\}$. Then the funding problem is solved if, having fixed the capital G_n at maturity *n*, in (8.7") we put $V_f(n) = G_n$.

If this funding is made by payments at the end of the period (*delayed payments*), it is enough to put $R_0 = 0$. Then the constraint between G_n that is to be set up in *n* and a vector $\{R_k\}$ of payments suitable for the funding is

$$G_n = \sum_{k=1}^{n-1} R_k \prod_{r=k+1}^{n} (1 + i_{r-1,r}) + R_n \tag{8.28}$$

Similarly if the sinking fund is accumulated with payments at the beginning of the period (*advance payments*), it is enough to put $R_n = 0$. Then the constraint by G_n in n and a vector $\{\ddot{R}_k\}$ of suitable payments is

$$G_n = \sum_{k=0}^{n-1} \ddot{R}_k \prod_{r=k}^{n-1} (1 + i_{r,r+1}) \tag{8.28'}$$

The accumulated capital sum at time $h<n$ with *delayed* payments is

$$M_h = G_h = \sum_{k=1}^{h-1} R_k \prod_{r=k+1}^{h} (1 + i_{r-1,r}) + R_h \tag{8.28''}$$

and, by advance payments, it is

$$M_h = G_h = \sum_{k=0}^{n-1} \ddot{R}_k \prod_{r=k}^{h-1} (1 + i_{r,r+1}) \tag{8.28'''}$$

For distinguishing the *principal shares* from *interest shares*, as $G_0 = 0$, in the *delayed* case such shares, denoted by C_h and I_h , are constrained by the system

$$(h = 1, \ldots, n) \begin{cases} C_h = G_h - G_{h-1} \\ I_h = G_{h-1}\ i_{h-1,h} \\ C_h = R_h + I_h \end{cases} \tag{8.29}$$

which implies the recursive equation

$$G_{h-1}(1+i_{h-1,h}) + R_h = G_h \tag{8.30}$$

that allows us to find (8.28) and (8.28'') again. In the *advance case*, denoting the principal repaid and interest paid with $\ddot{C}_h$ and $\ddot{I}_h$ and recalling (8.19), they are constrained by the system

$$(h = 0, \ldots, n-1) \begin{cases} \ddot{C}_h = G_{h+1} - G_h \\ \ddot{I}_h = G_{h+1}\ d_{h,h+1} \\ \ddot{C}_h = \ddot{R}_h + \ddot{I}_h \end{cases} \tag{8.29'}$$

which implies the recursive equation

$$G_h + R_h = G_{h+1}\,(1-d_{h,,h+1}) \tag{8.30'}$$

for which it is possible to find (8.28') and (8.28''') again.

If we consider *constant delayed payments R*, given as G_n and according to (8.28), they are obtained by

$$R = G_n \,/\, \left\{1 + \sum_{k=1}^{n-1} \prod_{r=k+1}^{n} (1+i_{r-1,r})\ \right\} \tag{8.31}$$

If we consider *constant advance payments*, denoting them by $\ddot{R}$ and according to (8.28'), the result is

$$\ddot{R} = G_n \,/\, \sum_{k=0}^{n-1} \prod_{r=k}^{n-1} (1+i_{r,r+1}) \tag{8.31'}$$

Exercise 8.5

Mr. John wishes to obtain €100.000 by annual constant payments in advance during 5 years, on a savings account yielding according to given forward rates. Let us calculate the constant payment and the sequence of balances.

A. The given rates are written in the 2nd column of the following *Excel* table to carry out the calculations. According to (8.31') the 3rd column allows us to calculate the constant payment $\ddot{R}$, which results in €17595.14. The 4th column gives the balances (i.e. the retro-reserves) at the end of each year.

	FUNDING IN ADVANCE DURING		5	YEARS	
Capital =	100000		Installment =	17595,14	
Year	Delayed forward rate	Accumulation factor	Retro-reserve	Interest paid	Principal repaid
k	$i_{k,k+1}$	$\Pi_{k,4}$	G_k	I_k	C_k
0	0.040	1.233106	0.00	703.81	18298.95
1	0.043	1.185679	18298.95	1543.45	19138.59
2	0.046	1.136797	37437.54	2531.50	20126.65
3	0.044	1.086804	57564.18	3307.01	20902.15
4	0.041	1.041000	78466.34	3938.52	21533.66
5		1.000000	100000.00		
Σ		5.683387		12024.29	100000.00

Table 8.9. *Example of funding in advance*

The *Excel* instructions are as follows. The first 5 rows devoted to data, titles and calculus of constant installments. D1: 5; B2: 100000; E2:= B2/C12; 6th to 11th rows:

column A (year): A6: 0; A7:= A6+1; copy A7, then paste on A8-A11;
column B (delayed forward rate): data input from B6 to B10;

column C (accum. fact. (k,4)):	C11: 1; C10:= C11*(1+B10); copy C10, then paste on C9 to C6;
column D (retro-reserve):	D6: 0; D7:= (D6+E$2)*(1+B6); copy D7, then paste on D8 to D10;
column E (interest paid):	E6:= (1-1/(1+B6))*D7; copy E6, then paste on E7 tE10;
column F (principal repaid):	F6:= E$2+E6; copy F6, then paste on F7 to F10.
row 12 (totals):	C12:= SUM(C6:C10); copy C12, then paste on E12 to F12.

Other cells: empty.

8.5. Valuations referred to shared loans in the term structure environment

In sections 6.8 and 6.9, all questions concerning the issue and management of bonds have been considered, from the point of view of the organization of the operation and of the valuation of reserves, usufructs and bare ownerships, with special reference to relations between bond prices and rates of return.

The previous investigation has been carried out assuming constant rates, both coupon rate and return rate. In this section we ought to complete this investigation in the term structure environment, supposing the structure to be assigned at an evaluation time put in 0, i.e. at bonds issue. For such structures, and assuming a coherent market, we shall use (7.25) and relations (7.26) to (7.39). As occurs for unshared loans amortization, when the change of rates can be performed over current time, the current uniperiods spot rates must be used by rules shown in section 6.9.4.

The treatment of the previous topics can be restricted in few words if we observe that a lot of schemes regarding bond management, shown in sections 6.8 and 6.9, is still valid in the new context. Indeed it is enough *to replace the constant coupon rate j with uniperiod forward coupon rates*, varying over the time interval, the structure of which we will denote by $\left\{j_{r-1,r}\right\}$. In addition, we will introduce, instead of only a valuation (or return) rate, an uniperiod forward return rate's structure, that we apply to give the value in 0 by discounting the cash-inflow subsequent to 0[8], or otherwise the price in 0 that assures the yield given by the given structure, that we denote $\left\{i_{r-1,r}\right\}$.

Unless stated otherwise the bonds have coupons, the period and payment times are annual and so are the rates. In case of semiannual coupons, it is sufficient to halve the coupon each year.

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8 The inverse problem, of calculating a balanced return structure according to purchase price, gives infinite solutions. Then it has a theoretic importance, linkable with Generalized Discounted Cash Flow (GDCF) questions seen in section 4.4.2.

8.5.1. *Financial flows by the issuer's and investor's point of view.*

Generalizing what was shown in section 6.8, we must distinguish the case of *only one maturity* for all bonds from that of *different maturities* with refunds according to a drawing plan:

a) Assumption of bonds with only one maturity

Let us recall some symbols specified in section 6.8, using:

- s = maturity (or life) of bonds, all issued in 0;
- c = redemption value of each bond (usually equal to par value);
- N = number of issued bonds;
- p_0 = purchase price of a bond at issue;
- p_r = purchase price of a bond at time r>0.

In addition, we use coupon $\{j_{r-1,r}\}$ and yield $\{i_{r-1,r}\}$ rate structure.

On the basis of such assumptions the parties make the following operations:

i) *issuer:* $(0,Np_0)\cup(1,-Ncj_{0,1})\cup\ldots\cup(n-1,-Ncj_{n-2,n-1})\cup(n,-Nc(1+j_{n-1,n}))$

ii) *buyer in* 0: $(0,-p_0)\cup(1,cj_{0,1})\cup\ldots\cup(n-1,cj_{n-2,n-1})\cup(n,c(1+j_{n-1,n}))$

iii) *buyer in r*: $(r,-p_r)\cup(r+1,cj_{r,r+1})\cup\ldots\cup(n-1,cj_{n-2,n-1})\cup(n,c(1+j_{n-1,n}))$

Such results hold under *annual coupons*. In the case of *semiannual coupons*, at k^{th} year for each bond we obtain two equal coupons whose amount is $cj_{k-1,k}/2$;

b) Assumption of different bonds maturities with refunds according to draw

Let n be the given loan time length with gradual refunds according to the following *drawing plan*

$$\{N_s\},\ \text{sub} \sum\nolimits_{s=1}^{n} N_s = N \tag{8.32}$$

In such a assumption the issuer is the debtor on a gradual amortization whereas the investors are creditors on an amortization with random time length and only one final refund after the payment of periodical interest. In detail, using (6.70) the operations are the following:

i) *issuer:* $(0,Np_0)\cup\left[\bigcup\nolimits_{s=1}^{n}(s,-N_sc-L_{s-1}cj_{s-1,s})\right]$

ii) *buyer in* 0 *with drawing and refund in s*>0:

$(0,-p_0)\cup(1,cj_{0,1})\cup\ldots\cup(s-1,cj_{s-2,s-1})\cup(s,c(1+j_{s-1,s}))$

iii) *buyer in r with drawing and refund in s>r*:

$(r,-p_r)\cup(r+1,cj_{r,r+1})\cup\ldots\cup(s-1,cj_{s-2,s-1})\cup(s,c(1+j_{s-1,s}))$

8.5.2. *Valuations of price and yield*

In section 6.9 valuations of bonds as a function of a given rate were performed; furthermore we have seen the correspondence between prices and discount rates that, given the prices, signify yield rates of the consequent investment operation.

We have to recall that, in a constant rate context, the correspondence between present values (or initial prices) and rates is biunique. On the other hand, in a varying rate context according to term structures the correspondence is only univocal, in the way that "term structure $\Rightarrow$ price", as soon as the bond loan parameters are assigned (see footnote 8). Then let us restrict ourselves, in this section devoted to valuations, to the calculation of the formula giving the balanced purchase price in the two schemes of loan management.

a) Assumption of bonds with only one maturity

Generalizing the results in section 6.9.2 and in (6.74) under coupon $\{j_{k-1,k}\}$ and return $\{i_{k-1,k}\}$ rates structures, with the symbols used in section 8.5.1 under a), the *purchase price* in 0 of bonds with life *s* is given by[9]

$$z_0^{(s)} = c\left[\sum\nolimits_{h=1}^{s} j_{h-1,h}\prod\nolimits_{k=1}^{h}(1+i_{k-1,k})^{-1} + \prod\nolimits_{k=1}^{s}(1+i_{k-1,k})^{-1}\right] \tag{8.33}$$

Furthermore, the *bond purchase price in r* (0<*r*<*s*), with unchanged term structures in (0,*s*) interval, is given by

$$z_r^{(s)} = c\left[\sum\nolimits_{h=r+1}^{s} j_{h-1,h}\prod\nolimits_{k=r+1}^{h}(1+i_{k-1,k})^{-1} + \prod\nolimits_{k=r+1}^{s}(1+i_{k-1,k})^{-1}\right] \tag{8.33'}$$

b) Assumption of drawing bonds

Generalizing the results of section 6.9.3 and (6.75') under coupon $\{j_{k-1,k}\}$ and yield $\{i_{k-1,k}\}$ rates structures, with the symbols used in section 8.5.1 under b) the *bond purchase price in* 0 is now the arithmetic mean, weighed by N_S , of bonds' prices having life *s*. Therefore it is worth

$$z_0 = \sum\nolimits_{s=1}^{n}\frac{N_s z^{(s)}}{N} = \sum\nolimits_{s=1}^{n}\frac{N_s\, c}{N}\left[\sum\nolimits_{h=1}^{s} j_{h-1,h}\prod\nolimits_{k=1}^{h}(1+i_{k-1,k})^{-1} + \prod\nolimits_{k=1}^{s}(1+i_{k-1,k})^{-1}\right] \tag{8.34}$$

9 Equation (8.33) shows that the inverse problem "price $\rightarrow$ structure $\{i_{k-1,k}\}$" gives infinite solutions of a difficult calculation in the generalized IRR environment.

Furthermore, the *bond purchase price in r* ($0<r<s$), with unchanged term structures in $(0,s)$ interval, is given by

$$z_r = \sum_{s=r+1}^{n} \frac{N_s\ c}{L_r} \left[\sum_{h=r+1}^{s} j_{h-1,h} \prod_{k=r+1}^{h} (1+i_{k-1,k})^{-1} + \prod_{k=r+1}^{s} (1+i_{k-1,k})^{-1} \right] \quad (8.34')$$